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RADIATION PATTERN, REACTIVE POWER,  
AND RESISTIVE APERTURE ANTENNAS

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of propagation) or orthogonal to it are related to the capacitive and inductive terms in the reactive power. It is also established that a circularly polarized pattern is necessarily associated with zero reactive power. More general conditions are then established and it is shown that zero reactive power is obtained when the plane wave spectrum (PWS) representing the aperture field has certain properties of rotational symmetry on the wave-number plane. An eigenvalue equation (defining those conditions) is established and the structure of the solution discussed. The physical antenna structures whose field can be represented by a PWS of the type studied are then identified with self-complementary apertures having rotational symmetry. Their inherent broadband properties are briefly discussed from the viewpoint of PWS theory.

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## Radiation Pattern, Reactive Power, and Resistive Aperture Antennas

### 1. INTRODUCTION

The radiative and reactive properties of a planar radiating aperture are closely interrelated. Once the radiation pattern (a complex vector function) is known in any arbitrary small angular region, the system of evanescent waves on the aperture plane and therefore the reactive power of the aperture, are, in principle, completely determined. This property is an immediate consequence of the analytical nature of the pattern (pointed out by Rhodes<sup>1</sup>), and has been exploited by several authors to establish expressions for the reactive power<sup>2,3</sup> after Woodward's pioneering work.<sup>4</sup>

A number of different expressions for the reactive power have been established, all of these having the common feature of consisting of an integral

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1. Rhodes, D.R. (1964) On a fundamental principle in the theory of planar antennas, Proc. IEEE 52:1013-1021.
2. Collin, R.E., and Rothschild, S. (1963) Reactive energy in aperture fields and aperture Q, Can. J. Phys. 41:1967-1969.
3. Borgiotti, G.V. (1963) Radiation and reactive energy of aperture antennas, IEEE Trans. on Antennas and Propagation, AP-11:94-95.
4. Woodward, P.M., and Lawson, J.E. (1948) The theoretical precision which an arbitrary radiation pattern may be obtained from a source of finite size, J. IEE(London) 95(Pt. 111):363-370.

(extended to the "Invisible Space" of the wavenumber plane; that is, the region whose points represent evanescent waves) of a function strictly related to the analytic continuation of the radiation pattern.<sup>1-3</sup> Among them a particularly simple expression for the reactive power consists of the difference of two positive integrals, each depending only upon the transverse electric (TE) and transverse magnetic (TM) parts of the field with respect to the normal to the aperture.<sup>5</sup> It was later recognized that the components of the radiation field polarized in the plane of incidence and orthogonal to it, give rise to the inductive and capacitive terms, respectively, in the expression of the reactive power.<sup>6</sup> Although this led to an insight into the close relationship clearly existing between aperture reactance and polarization of the radiation pattern, no systematic investigation of the question was attempted.

The main purpose of this report is to establish certain structural properties of a peculiar class of Plane Wave Spectra (PWS) representing an aperture field (or, equivalently of radiation patterns) which make the aperture reactive power equal to zero. To achieve this objective the entire question of the representation of an aperture field through a PWS is reexamined from a novel viewpoint. The Fourier Transform (FT) of the transverse electric field on the aperture is considered as a two-dimensional vector field on the wavenumber plane, represented through two scalar functions, which are its coordinates in a chosen vector basis (Section 3). The choice of the latter has an important physical significance since it corresponds to the decomposition of the PWS into two components having differently polarized radiation patterns and different reactive properties. In Section 4, it is shown that the conductance and the susceptance of an aperture can be expressed as integrals (extended over the visible and invisible space) of two quadratic forms (whose explicit expressions depend upon the basis chosen to represent the PWS). The diagonal and off diagonal terms of each of the  $2 \times 2$  characteristic matrices of the quadratic forms, represent the self and cross contributions of the two components of the PWS to the conductance or the susceptance. By using this technique it is shown that, if the radiation pattern is decomposed into two components circularly polarized at infinity, cross terms only contribute to the aperture susceptance. It is thus established that an aperture whose radiation pattern is circularly polarized necessarily has a reactive power equal to zero.

A different and more general structural requirement for the PWS having zero reactive power is then introduced. It consists essentially in requiring that

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5. Borgiotti, G.V. (1967) On the reactive energy of an aperture, IEEE Trans. on Antennas and Propagation, AP-15:565-569.
  6. Borgiotti, G.V. (1968) A novel expression for the mutual admittance of planar radiating elements, IEEE Trans. on Antennas and Propagation, AP-16(No. 3):329-333.

the function representing the density of reactive power per unit of area of the wavenumber plane, be azimuthally periodic around the origin and take equal positive and negative values. This simple requirement leads to a field whose properties are recognized to be those associated with rotationally symmetric self complementary structures, whose inherent broadband properties are discussed from the viewpoint of PWS theory.

Throughout this report, the arguments of the various functions will be deleted in most cases, and retained only when it is felt necessary for reasons of clarity. Also, a "real type" scalar product is used; that is, the operation of complex conjugation is always explicitly indicated.

## 2. BACKGROUND AND NOTATIONS

Let the aperture A be cut on a perfectly conducting ground plane. The apertures we will consider are not necessarily "strictly limited" in extent, in order not to rule out from our considerations certain structures (like the infinite spiral for example), which are idealized and mathematically tractable models of real world (that is, finite) antennas. We will require, however, that the transverse electric field on the aperture be square integrable. This excludes, from this treatment, periodic structures like periodic arrays of apertures. However, with minor conceptual and formal modifications (namely use of Fourier Series rather than integrals) the latter case could also be accommodated.

Let  $z = 0$  be the aperture plane, in the geometrical frame of reference  $x, y, z$ . The origin is located at a point of the aperture which will coincide with its center for those apertures having point symmetry. The axis  $z$  points towards the half-space of radiation. Denote a vector position on the aperture plane by:

$$\underline{x} = x\hat{x} + y\hat{y} . \quad (1)$$

To complete the geometrical description let  $r, \theta, \phi$  be a spherical system (associated in a standard way with the  $x, y, z$  axes) with  $\hat{\theta}$  and  $\hat{\phi}$  unit vectors in  $\theta$  and  $\phi$  directions. Let  $\alpha$  and  $\beta$  be the cosines of the direction of observation with respect to the  $x, y$  axes. Then:

$$\alpha = \sin \theta \cos \phi \quad (2)$$

$$\beta = \sin \theta \sin \phi . \quad (3)$$



The tangential magnetic and electric field on  $x = 0$  will be denoted by  $\underline{H}_t(x)$  and  $\underline{E}_t(x)$ , the latter differing from zero only on points of the aperture A. The FT of  $\underline{E}_t(x)$  is defined as follows:

$$\underline{\mathcal{E}}_t(\underline{u}) = \frac{1}{2\pi} \iint_A \underline{E}_t(\underline{x}) e^{j\underline{u}\underline{x}} d^2\underline{x} \quad (4)$$

where

$$\underline{u} = u\hat{x} + v\hat{y} \quad (5)$$

is a position vector in the plane  $u, v$  of the wavenumbers in directions  $x$  and  $y$ . The element of area on the aperture plane has been denoted by  $d^2\underline{x}$ . It is convenient to introduce (besides the cartesian components  $u, v$ ) a polar reference system on the wavenumber plane with radius vector  $t$ :

$$t = |\underline{u}| = \sqrt{u^2 + v^2} \quad (6)$$

and argument  $\mu$ :

$$\cos \mu = \frac{u}{t} \quad \sin \mu = \frac{v}{t}. \quad (7)$$

A unit vector in the wavenumber plane pointing in the radial direction will be denoted by

$$\hat{\rho}(\underline{u}) = t^{-1} \underline{u} \quad (8)$$

and a unit vector in the circumferential direction by

$$\hat{\psi}(\underline{u}) = \hat{z} \times \hat{\rho}(\underline{u}). \quad (9)$$

If  $w$  is the propagation constant in  $z$  direction, then the propagation vector  $\underline{k}$  is defined as

$$\underline{k} = \underline{u} + w\hat{z}. \quad (10)$$

If  $k = 2\pi/\lambda$  is the free space propagation constant, then  $w$  is related to  $u, v$  by the dispersion relationship. Thus

$$w = \left| \sqrt{k^2 - u^2 - v^2} \right| \quad (11)$$

in the "Visible Space", that is, the set  $C$  of points in the wavenumber plane such that

$$|\underline{u}| \leq k. \quad (12)$$

In the "invisible Space", that is, the set  $\bar{C}$  of points for which

$$|\underline{u}| > k, \quad (13)$$

it is

$$w = -j \sqrt{k^2 - u^2 - v^2}. \quad (14)$$

The choice of the sign in Eqs. (11) and (14) is obviously dictated by radiation conditions.

The FT of the transverse magnetic field  $\underline{H}_t(\underline{x})$  on the aperture plane will be denoted by  $\underline{\mathcal{E}}_t(\underline{u})$ . Because of Maxwell's equations,  $\underline{H}_t(\underline{x})$  is of course uniquely determined once  $\underline{E}_t(\underline{x})$  is assigned.

### 3. PWS REPRESENTATIONS OF THE EM FIELD

#### 3.1 General Formulation

It is of course well known that every electromagnetic field can be represented by two scalar functions in conjunction with suitable vector differential operators. It is also known that while the representation is not unique, different representations are related through linear transformations.

This fundamental property of the EM field can be conveniently rephrased in terms of linear spaces. In fact its geometrical interpretation in the framework of the present treatment, consists simply of the obvious possibility of representing the two-dimensional vector function  $\underline{\mathcal{E}}_t(\underline{u})$  on the wavenumber plane in different coordinate systems (or, in linear spaces terminology, vector bases). Generally,  $\underline{\mathcal{E}}_t(\underline{u})$  can be expressed as follows:

$$\underline{\mathcal{E}}_t(\underline{u}) = c_1(\underline{u})\underline{s}_1(\underline{u}) + c_2(\underline{u})\underline{s}_2(\underline{u}) \quad (15)$$

where the two scalars  $c_1(\underline{u})$ ,  $c_2(\underline{u})$  (coordinates of  $\underline{\mathcal{E}}_t(\underline{u})$  in the vector basis  $\underline{s}_1(\underline{u})$ ,  $\underline{s}_2(\underline{u})$ ) can be considered as the components of a column vector  $\underline{c}(\underline{u})$ :

$$\underline{c}(\underline{u}) = \begin{bmatrix} c_1(\underline{u}) \\ c_2(\underline{u}) \end{bmatrix}. \quad (16)$$

In the following, it will prove convenient to express the radiation pattern corresponding to the PWS Eq. (15) as the weighted sum of two partial patterns  $\vec{F}_{s1}(\theta, \phi)$  and  $\vec{F}_{s2}(\theta, \phi)$  obtained by assuming in (15)  $c_1(\underline{u}) = 1$ ,  $c_2(\underline{u}) = 0$  and  $c_1(\underline{u}) = 0$ ,  $c_2(\underline{u}) = 1$  respectively. As discussed in Appendix A, the total pattern takes the form:

$$\vec{F}(\theta, \phi) = c_1(k\alpha, k\beta) \vec{F}_{s1}(\theta, \phi) + c_2(k\alpha, k\beta) \vec{F}_{s2}(\theta, \phi). \quad (17)$$

Thus the scalars  $c_1(\underline{u})$ ,  $c_2(\underline{u})$  are related to the complex amplitudes of the partial patterns and the vectors  $\underline{s}_1(\underline{u})$ ,  $\underline{s}_2(\underline{u})$  to their polarization properties.

We will proceed now to discuss a few examples of different PWS representations (among the infinite number possible).

### 3.2 TE and TM Field Components

Let the vector basis used for representing the field be:

$$\underline{s}_1(\underline{u}) = \hat{\rho}(\underline{u}) \quad \underline{s}_2(\underline{u}) = \hat{\psi}(\underline{u}). \quad (18)$$

Thus:

$$\underline{\mathcal{E}}_t(\underline{u}) = \underline{\mathcal{E}}_\rho(\underline{u})\hat{\rho} + \underline{\mathcal{E}}_\psi(\underline{u})\hat{\psi}(\underline{u}) \quad (19)$$

which is, of course, in the general form of Eq. (15). The representation (19) of  $\underline{\mathcal{E}}_t(\underline{u})$  in polar form corresponds to the decomposition of the total (that is, radiative plus evanescent) field into two noninteracting components;  $\underline{\mathcal{E}}_\rho(\underline{u})$  being related to the transverse magnetic (TM) and  $\underline{\mathcal{E}}_\psi(\underline{u})$  to the transverse electric (TE) parts of the field with respect to the direction orthogonal to the aperture plane, respectively. This property has been discussed briefly in.<sup>7</sup> The two partial patterns are

$$\vec{F}_\rho(\theta, \phi) = \hat{\theta} \quad (20)$$

7. Borgiotti, G. V. (1968) Model analysis of periodic phased arrays of apertures, Proc. IEEE, 56:1881-1892 (Appendix 1).

$$\vec{F}_\phi(\theta, \phi) = \cos \theta \hat{\phi} \quad (21)$$

(a result published already in a different form, in<sup>6</sup>). Thus the radial component of  $\underline{\mathcal{E}}_t(\underline{u})$  gives rise to a radiation field polarized in the plane of incidence and the circumferential component to a field polarized in direction orthogonal to it. For future reference, we explicitly write the total pattern which from (17) and (19) to (21) is:

$$\vec{F}(\theta, \phi) = \mathcal{E}_\rho(k\alpha, k\beta) \hat{\theta} + \cos \theta \mathcal{E}_\psi(k\alpha, k\beta) \hat{\phi}. \quad (22)$$

There is one point to be noticed about the field representations (19). Since the vector function  $\underline{\mathcal{E}}_t$  is continuous at  $\underline{u} = 0$ , the scalar functions  $\mathcal{E}_\rho$  and  $\mathcal{E}_\psi$  must either necessarily have a discontinuity at that point or must be zero. It is also easily seen that for the same reasons, a field purely TE or TM with respect to  $z$  must necessarily have zero radiation at broadside; that is, the single scalar function representing it must have a zero for  $\underline{u} = 0$ .

### 3.3 Rectangular Components

For completeness here the representation through rectangular components is included, since it has been widely used in previous related work:<sup>1, 8</sup>

$$\underline{s}_1(\underline{u}) = \hat{x} \quad \underline{s}_2(\underline{u}) = \hat{y}; \quad (23)$$

that is

$$\underline{\mathcal{E}}_t(\underline{u}) = \mathcal{E}_x(\underline{u})\hat{x} + \mathcal{E}_y(\underline{u})\hat{y}. \quad (24)$$

In this case

$$\vec{F}_x(\theta, \phi) = \cos \phi \hat{\theta} - \cos \theta \sin \phi \hat{\phi} \quad (25)$$

$$\vec{F}_y(\theta, \phi) = \sin \phi \hat{\theta} + \cos \theta \cos \phi \hat{\phi}. \quad (26)$$

We will not use this representation in the sequel.

8. Borgiotti, G.V. (1963) Fourier transforms method in aperture antennas problems, Alta Frequenza, 32(No. 11):808-816.

### 3.4 Field Components Circularly Polarized at Infinity

Let the PWS be represented as:

$$\underline{\mathcal{E}}_t(\underline{u}) = \underline{\mathcal{E}}_R(\underline{u}) \underline{R}(\underline{u}) + \underline{\mathcal{E}}_L(\underline{u}) \underline{L}(\underline{u}) \quad (27)$$

with

$$\underline{R}(\underline{u}) = \hat{\rho} + j \frac{k}{w} \hat{\Psi} \quad \underline{L}(\underline{u}) = \hat{\rho} - j \frac{k}{w} \hat{\Psi} \quad (28)$$

By applying Eq. (22) it is apparent that each single component field is circularly polarized at infinity, in the clockwise and counterclockwise directions respectively.

## 4. COMPLEX POWER AND PATTERN POLARIZATION

### 4.1 Conductance and Susceptance Matrices

The complex power associated with the aperture is obtained by applying Poynting's theorem for sinusoidal fields to the region limited by the aperture plane and a hemisphere of radius tending to infinity (in  $z > 0$ ). The active and reactive powers are the real and imaginary parts of the integral:

$$P = P_r + jP_j = \frac{1}{2} \iint_A \underline{E}_t(\underline{x}) \times \underline{H}_t^*(\underline{x}) \cdot \hat{z} d^2 \underline{x} \quad (29)$$

where  $d^2 \underline{x}$  indicates the element of area in the aperture plane and the star denotes complex conjugate. By using Parseval's theorem, Eq. (29) becomes<sup>3</sup>

$$P_r + jP_j = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \underline{\mathcal{E}}_t(\underline{u}) \times \underline{\mathcal{H}}_t^*(\underline{u}) \cdot \hat{z} d^2 \underline{u} \quad (30)$$

where  $d^2 \underline{u}$  is the elementary area on the wavenumber plane. In Appendix B, it is shown that the integrand of (30) through some manipulations can be put in the form:

$$\underline{\mathcal{E}}_t \times \underline{\mathcal{H}}_t^* \cdot \hat{z} = \frac{1}{k\eta w} \left[ k^2 (\underline{\mathcal{E}}_t^+ \cdot \underline{\mathcal{E}}_t) - t^2 (\underline{\mathcal{E}}_t^+ \cdot \hat{\Psi})(\hat{\Psi} \cdot \underline{\mathcal{E}}_t) \right] \quad (31)$$

$\eta$  being the intrinsic impedance of the vacuum and the cross indicating the conjugate transpose. In (31) the argument of the functions have not been explicitly indicated. From now on they will be dropped in most cases to simplify notations. Expression (31) can also be written concisely as

$$\underline{\underline{\mathcal{E}}}_t \times \underline{\underline{\mathcal{H}}}_t^* \cdot \hat{\underline{\underline{z}}} = \underline{\underline{\mathcal{E}}}_t^+ \cdot \underline{\underline{Y}}^+ \underline{\underline{\mathcal{E}}}_t \quad (32)$$

where  $\underline{\underline{Y}}$  is identified with the dyadic operator

$$\underline{\underline{Y}} = \frac{1}{k \eta w} \left[ k^2 \underline{\underline{1}} - \hat{\underline{\underline{\Psi}}} > t^2 < \hat{\underline{\underline{\Psi}}} \right] \quad (33)$$

where the convenient bracket notation has been used.  $\underline{\underline{1}}$  is the second order unit dyad.

The operator  $\underline{\underline{Y}}$  is symmetric but not Hermitian. However, it can be decomposed into two Hermitian (in fact real symmetric) parts  $\underline{\underline{G}}$  and  $\underline{\underline{B}}$

$$\underline{\underline{Y}} = \underline{\underline{G}} + j \underline{\underline{B}} \quad (34)$$

where

$$\underline{\underline{G}} = \frac{1}{2} (\underline{\underline{Y}} + \underline{\underline{Y}}^+) \quad (35)$$

and

$$\underline{\underline{B}} = \frac{1}{2j} (\underline{\underline{Y}} - \underline{\underline{Y}}^+) . \quad (36)$$

From the inspection of (33), by recalling (11) to (14), it is apparent that the operators  $\underline{\underline{G}}$  and  $\underline{\underline{B}}$  (conventionally called "Conductance" and "Susceptance") are different from zero only in the visible and invisible space, respectively. Thus the real and reactive powers are expressed through integrals of Hermitian quadratic forms:

$$P_R = \frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \underline{\underline{\mathcal{E}}}_t^+ \cdot \underline{\underline{G}} \underline{\underline{\mathcal{E}}}_t d^2 \underline{\underline{u}} \quad (37)$$

and

$$P_j = -\frac{1}{2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \underline{\underline{\mathcal{E}}}_t^+ \cdot \underline{\underline{B}} \underline{\underline{\mathcal{E}}}_t d^2 \underline{\underline{u}} . \quad (38)$$

The quadratic forms in (37) and (38) can be represented through matrices whose explicit expression depend upon the vector basis used.

Let  $\underline{\underline{e}}_t(u)$  be represented in the  $\underline{s}_1, \underline{s}_2$  basis, through the column vector (16). Then

$$\underline{\underline{e}}_t^+ \cdot \underline{\underline{G}} \underline{\underline{e}}_t = \underline{c}^+ \cdot \underline{\underline{G}}^{(s)} \underline{c} \quad (39)$$

where  $\underline{\underline{G}}^{(s)}$  is the matrix representation of the conductance in the chosen basis:

$$\underline{\underline{G}}^{(s)} = \begin{bmatrix} \underline{s}_1^+ \cdot \underline{\underline{G}} \underline{s}_1 & \underline{s}_2^+ \cdot \underline{\underline{G}} \underline{s}_1 \\ \underline{s}_2^+ \cdot \underline{\underline{G}} \underline{s}_1 & \underline{s}_2^+ \cdot \underline{\underline{G}} \underline{s}_2 \end{bmatrix} \quad (40)$$

and similarly

$$\underline{\underline{e}}_t^+ \cdot \underline{\underline{B}} \underline{\underline{e}}_t = \underline{c}^+ \cdot \underline{\underline{B}}^{(s)} \underline{c} \quad (41)$$

with  $\underline{\underline{B}}^{(s)}$  denoting the matrix representation of  $\underline{\underline{B}}$ .

The reason for the development of the above formalism stems from the physical significance of the various terms in the expressions of  $\underline{\underline{G}}^{(s)}$  and  $\underline{\underline{B}}^{(s)}$ .

In fact, a term

$$\langle \underline{s}_i^+ \cdot \underline{\underline{G}} \underline{s}_i \rangle \quad (i = 1, 2) \quad (42)$$

or

$$\langle \underline{s}_i^+ \cdot \underline{\underline{B}} \underline{s}_i \rangle \quad (i = 1, 2) \quad (43)$$

represents the active or reactive power due to the polarization components  $\underline{s}_i$  when acting alone. Instead the cross forms

$$\langle \underline{s}_i^+ \cdot \underline{\underline{G}} \underline{s}_k \rangle \quad (i \neq k = 1, 2) \quad (44)$$

$$\langle \underline{s}_i^+ \cdot \underline{\underline{B}} \underline{s}_k \rangle \quad (i \neq k = 1, 2) \quad (45)$$

represent the effect of the interaction of two different polarization components on the active and reactive powers. The theory will not be applied to two PWS representation of particular importance.

#### 4.2 $\theta$ and $\phi$ Partial Patterns

Let the field be represented as in Section 3.2. It is promptly recognized that  $\hat{\rho}$  and  $\hat{\psi}$  are eigenvectors of the conductance and susceptance operators. With such a basis, the matrices representing Eqs. (35) and (36) take the diagonal forms

$$\underline{\underline{G}}(\rho, \psi) = \frac{1}{2} \begin{bmatrix} \frac{k}{\eta} \left( \frac{1}{w} + \frac{1}{w^*} \right) & 0 \\ 0 & \frac{1}{k\eta} (w + w^*) \end{bmatrix} \quad (46)$$

and

$$\underline{\underline{B}}(\rho, \psi) = \frac{1}{2} \begin{bmatrix} \frac{k}{\eta} \left( \frac{1}{w} - \frac{1}{w^*} \right) & 0 \\ 0 & \frac{1}{k\eta} (w - w^*) \end{bmatrix}. \quad (47)$$

Thus, from (37):

$$P_R = \frac{1}{2\eta} \iint_C \left( \frac{w}{k} \mathcal{E}_\psi \mathcal{E}_\psi^* + \frac{k}{w} \mathcal{E}_\rho \mathcal{E}_\rho^* \right) d^2 \underline{u} \quad (48)$$

$$P_J = \frac{1}{2\eta} \iint_{\mathcal{E}^C} \left( \frac{|w|}{k} \mathcal{E}_\psi \mathcal{E}_\psi^* - \frac{k}{|w|} \mathcal{E}_\rho \mathcal{E}_\rho^* \right) d^2 \underline{u} \quad (49)$$

The lack of off diagonal terms in (46) and (47) implies the cross terms of the type (44) and (45) are zero. The TM (that is,  $\mathcal{E}_\rho$ ) and the TE (that is,  $\mathcal{E}_\psi$ ) components of the field give rise not only to orthogonal radiation patterns as expressed by (22) and (44) but also to orthogonal evanescent fields, with reactive power of capacitive and inductive types. On the basis of the previous treatment, certain properties of the aperture field can be summarized by:



Proposition I: If in a certain arbitrary solid angle the radiation pattern is polarized in the planes of incidence (that is,  $F_\phi = 0$ ), then  $\mathcal{E}_\psi$  equals zero in the entire wavenumber plane and the aperture reactance is purely capacitive. Similarly, if in any angular region the radiation pattern is polarized in directions orthogonal to incidence plane, the aperture reactance is purely inductive. In both the cases, the radiation pattern has a zero at broadside.

#### 4.3 Circularly Polarized Partial Patterns

Suppose now that the field is represented through circularly polarized components at infinity as expressed by Eq. (27). Then:

$$\underline{\underline{G}}^{(R, L)} = \begin{bmatrix} \frac{k}{\eta} \left( \frac{1}{w} + \frac{1}{w^*} \right) & 0 \\ 0 & \frac{k}{\eta} \left( \frac{1}{w} + \frac{1}{w^*} \right) \end{bmatrix}. \quad (50)$$

However, the vectors  $\underline{R}$  and  $\underline{L}$  do not diagonalize the susceptance operator. In fact, in the representation of the susceptance only off diagonal terms of the type (45) are present:

$$\underline{\underline{B}}^{(R, L)} = \begin{bmatrix} 0 & \frac{k}{j\eta} \frac{w^* - w}{ww^*} \\ \frac{k}{j\eta} \frac{w^* - w}{ww^*} & 0 \end{bmatrix}. \quad (51)$$

Thus the radiated power is from (37) and (42):

$$P_r = \frac{k}{\eta} \iint_C \left[ |\mathcal{E}_R|^2 + |\mathcal{E}_L|^2 \right] w d^2 \underline{u} \quad (52)$$

that is, it is the sum of the powers associated with left and right polarizations. The reactive power is from (36) and (45):

$$P_j = -\frac{k}{\eta} \iint_C \left( \mathcal{E}_R^* \mathcal{E}_L + \mathcal{E}_R \mathcal{E}_L^* \right) |w| d^2 \underline{u} \quad (53)$$

and is thus due to the interactions of the two systems of evanescent waves associated with the two circularly polarized components of the aperture field.

From (53) the property expressed by Proposition II follows: If the radiation pattern is (exactly) circularly polarized in any angular region (and consequently in the entire hemisphere at infinity), then the reactive power is necessarily zero.

The above sufficient condition for zero reactance is a very restrictive one (and, of course, is by no means necessary). The kind of structure that can support a field perfectly circularly polarized at infinity, is a question that will be postponed until the end of Section 7.

## 5. "GLOBAL" RESONANCE AND RESISTIVE PWS

The condition of zero reactive power, written here as follows:

$$\int_k^\infty t dt \int_0^{2\pi} d\mu \underline{\mathcal{E}}_t^+ \underline{B} \underline{\mathcal{E}}_t = 0 \quad (54)$$

expresses in general only a "global" resonance condition, that is (as can be seen from Eq. (49)) the fact that the inductive and capacitive powers have the same magnitude and opposite sign. The occurrence of such a circumstance can be expected to be strongly frequency dependent.<sup>1, 2</sup> A more restrictive condition of "local" character will now be introduced leading to the definition of a class of PWS with zero associated reactive power, independent of frequency (in a sense later discussed).

To simplify the developments in the sequel, the operator  $R(\gamma)$  is introduced whose effect on a scalar function  $f(t, \mu)$  is that of rotating it by an angle  $\gamma$  around the origin of the wavenumber plane:

$$R(\gamma) f(t, \mu) = f(t, \mu - \gamma) . \quad (55)$$

If  $R(\gamma)$  is applied to a two-dimensional vector (like  $\underline{\mathcal{E}}_t$ ), it acts on each component in the way expressed by (55).

We will now require the following equality to hold for every point of the invisible space and for a certain fixed  $\gamma$ :

$$\underline{\mathcal{E}}_t^*(t, \mu) \cdot \underline{B} \underline{\mathcal{E}}_t(t, \mu) = - \left[ \underline{R}(\gamma) \underline{\mathcal{E}}_t(t, \mu) \right]^+ \cdot \underline{B} \left[ \underline{R}(\gamma) \underline{\mathcal{E}}_t(t, \mu) \right] . \quad (56)$$

The significance of (56) is the following: For every evanescent component wave propagating into direction  $\mu$  (with phase velocity  $k/t$ <sup>8</sup>) and contributing a certain elementary amount to the reactive power, there exists another evanescent wave,

propagating (with the same velocity) in the direction  $\mu - \gamma$  and contributing an opposite elementary amount to the reactive power. A PWS for which (56) holds will be called "resistive" since (54) is evidently satisfied by it.

Since, obviously, it is identically

$$\underline{\underline{\mathcal{E}}}_t^+ \cdot \underline{\underline{B}} \underline{\underline{\mathcal{E}}}_t = \left[ \underline{\underline{R}}(2\pi) \underline{\underline{\mathcal{E}}}_t \right]^+ \cdot \underline{\underline{B}} \left[ \underline{\underline{R}}(2\pi) \underline{\underline{\mathcal{E}}}_t \right] \quad (57)$$

from (56) and (57), it follows that  $\gamma$  cannot be arbitrary but must be

$$\gamma = \frac{\pi}{N} \quad (58)$$

with  $N$  a positive integer. From (56) and (58) and (34) to (36) and the analyticity of the various functions involved, the other condition for  $\underline{\underline{\mathcal{E}}}_t$  follows:

$$\underline{\underline{\mathcal{E}}}_t^+ \underline{\underline{Y}} \underline{\underline{\mathcal{E}}}_t = \left\{ \left[ \underline{\underline{R}}\left(\frac{\pi}{N}\right) \underline{\underline{\mathcal{E}}}_t \right]^+ \cdot \underline{\underline{Y}} \left[ \underline{\underline{R}}\left(\frac{\pi}{N}\right) \underline{\underline{\mathcal{E}}}_t \right] \right\}^* . \quad (59)$$

Equations (56) and (59) imply that for resistive PWS, the density of active power on the wavenumber plane is a periodic function of the azimuthal coordinate with periodicity  $\pi/N$ , as is also the power radiation pattern. The density of reactive power is instead a periodic function of periodicity  $2\pi/N$ , taking opposite values at points azimuthally distant by  $\pi/N$ .

## 6. THE ANALYTICAL STRUCTURE OF RESISTIVE PWS

We rewrite Eq. (59) as follows (by using bracket notation, which helps the clarity of presentation):

$$\langle \underline{\underline{\mathcal{E}}}_t^+ \underline{\underline{Y}} \underline{\underline{\mathcal{E}}}_t \rangle = \langle (\underline{\underline{Y}} \underline{\underline{\mathcal{E}}}_t)^+ (\underline{\underline{R}} \underline{\underline{\mathcal{E}}}_t) \rangle . \quad (60)$$

If  $\underline{\underline{Q}}$  is a linear operator, then (60) is satisfied if

$$\underline{\underline{\mathcal{E}}}_t = \underline{\underline{Q}} \underline{\underline{Y}} \underline{\underline{R}} \underline{\underline{\mathcal{E}}}_t \quad (61)$$

$$\underline{\underline{Y}} \underline{\underline{\mathcal{E}}}_t = (\underline{\underline{Q}}^+)^{-1} \underline{\underline{R}} \underline{\underline{\mathcal{E}}}_t . \quad (62)$$

Equations (61) and (62) are obtained by requiring the two scalar products on the right and left side of (60) to be equal. The following matrix equation for  $\underline{\underline{Q}}$  follows:

$$\underline{\underline{Q}} \underline{\underline{Y}} = \underline{\underline{Y}}^{-1} (\underline{\underline{Q}}^+)^{-1} . \quad (63)$$

For any choice of an orthogonal basis,  $\underline{\underline{Q}}$  is easily found to be of the form:

$$\underline{\underline{Q}} = e^{j\tau} \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \quad (64)$$

with  $\tau$  real. In (64), either the upper or the lower signs should be taken together. By introducing (64) into (61), the following homogeneous eigenvalue equation is obtained:

$$\underline{\underline{E}}_t e^{-j\tau} = \frac{1}{\eta} \underline{\underline{Y}}^{-1} \begin{bmatrix} 0 & \mp \underline{\underline{R}} \left( \frac{\pi}{N} \right) \\ \pm \underline{\underline{R}} \left( \frac{\pi}{N} \right) & 0 \end{bmatrix} \underline{\underline{E}}_t \quad (65)$$

(and, of course, the same equation would be obtained from (62)). We may summarize the results obtained so far in Proposition III: A PWS is resistive (in the sense expressed by (56)) if and only if the FT of the transverse electric field satisfies the eigenvalue equation (65).

In polar coordinates, (65) is explicitly written:

$$e^{-j\tau} \begin{bmatrix} \underline{\underline{E}}_\rho \\ \underline{\underline{E}}_\psi \end{bmatrix} = \pm \begin{bmatrix} \frac{w}{k} & 0 \\ 0 & \frac{k}{w} \end{bmatrix} \begin{bmatrix} 0 & -\underline{\underline{R}} \left( \frac{\pi}{N} \right) \\ \underline{\underline{R}} \left( \frac{\pi}{N} \right) & 0 \end{bmatrix} \begin{bmatrix} \underline{\underline{E}}_\rho \\ \underline{\underline{E}}_\psi \end{bmatrix} \quad (66)$$

equivalent to the pair of relationships:

$$e^{-j\tau} \underline{\underline{E}}_\rho = \pm \frac{w}{k} \underline{\underline{R}} \left( \frac{\pi}{N} \right) \underline{\underline{E}}_\psi \quad (67)$$

$$e^{-j\tau} \underline{\underline{E}}_\psi = \mp \frac{k}{w} \underline{\underline{R}} \left( \frac{\pi}{N} \right) \underline{\underline{E}}_\rho . \quad (68)$$

By combining (67) and (68), it follows that  $\underline{\mathcal{E}}_\rho$  and  $e^{-j2\tau}$  are an eigenfunction and an eigenvalue of the equation:

$$-e^{-j2\tau} \underline{\mathcal{E}}_\rho = \underline{\underline{R}}^2 \left( \frac{\pi}{N} \right) \underline{\mathcal{E}}_\rho \quad (69)$$

and an identical equation is obtained for  $\underline{\mathcal{E}}_\psi$ . Since

$$\underline{\underline{R}}^{2N} \left( \frac{\pi}{N} \right) = \underline{\underline{R}}(2\pi) = \underline{\underline{U}} \quad (70)$$

with  $\underline{\underline{U}}$  the identity operator, the eigenvalues of  $\underline{\underline{R}}^2 \left( \frac{\pi}{N} \right)$  satisfy the equation:

$$(-e^{-j2\tau_s}) \underline{\underline{N}} = 1, \quad (s = 0, 1 \dots N-1) \quad (71)$$

that is, they consist of the set:

$$-e^{-j2\tau_s} = e^{-j\frac{2\pi s}{N}} \quad (s = 0, 1 \dots N-1) \quad (72)$$

The eigenfunctions of (69) are promptly found to have the general form:

$$\underline{\mathcal{E}}_\rho^{(s)}(t, \mu) = \sum_{q=-\infty}^{+\infty} a_{s+qN}(t) e^{j(s+qN)\mu} \quad (s = 0, 1 \dots N-1) \quad (73)$$

Thus  $\underline{\mathcal{E}}_\rho^{(s)}$  is the product of a periodic function (of period  $2\pi/N$ ) by a linear phase term (of period  $2\pi/s$ ). The circumferential component  $\underline{\mathcal{E}}_t$  has of course the same type of angular dependence. From (72),

$$e^{-j\tau_s} = \pm j e^{-j\frac{\pi s}{N}} \quad (74)$$

Each of the  $2N$  values (74) corresponds to a different eigensolution of (66). Since the choices of the signs in (64) and (68) are independent, the latter equation gives:

$$\underline{\mathcal{E}}_\psi^{(s)}(t, \mu) = \pm j \frac{k}{w} \sum_{q=-\infty}^{+\infty} a_{s+qN}(t) e^{j(s+qN)\mu} (-1)^q \quad (s=0, 1 \dots N-1) \quad (75)$$

The radiation pattern from (22) is found to be

$$\vec{F}(\theta, \phi) = \sum_{q=-\infty}^{+\infty} a_{s+qN} (k \sin \theta) e^{j(s+qN)\mu} (\hat{\theta} \pm j(-1)^q \hat{\phi}). \quad (76)$$

Suppose now that every harmonic with  $s+qN \neq n$  is negligible in (76). Some remarks on such an hypothesis for practical structure will be made in Section 7. Then:

$$\vec{F}(\theta, \phi) = a_n (k \sin \theta) e^{jn\phi} (\hat{\theta} \pm j \hat{\phi}) \quad (77)$$

a circularly polarized pattern. It could be shown that if the radiation pattern does not have a zero at broadside and  $n = +1$ , then the upper sign in (77) must be chosen (vice versa if  $n = -1$ ).

If the order  $N$  of rotational symmetry goes to infinity, (65) becomes:

$$e^{-j\tau} \underline{\mathcal{E}}_t = \frac{1}{\eta} \begin{bmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{bmatrix} \underline{\mathcal{E}}_t \quad (78)$$

yielding the following relationship (for any  $s$ )

$$\underline{\mathcal{E}}_\psi = \pm j \frac{k}{w} \underline{\mathcal{E}}_\rho \quad (79)$$

corresponding to a circularly polarized pattern.

## 7. SELF-COMPLEMENTARY STRUCTURES

In this section certain features of self-complementary antennas (for example, the various types of spiral structures) will be expressed in terms of the resistive PWS concept developed in the previous sections. In fact it will be recognized that the electromagnetic fields of this type of antenna can be represented through this peculiar type of PWS.

We notice first that Eq. (65) by using the FT of Maxwell's equations can be transformed into:

$$\underline{\mathcal{E}}_t = \pm R \left( \frac{\pi}{N} \right) \eta \underline{\mathcal{X}}_t e^{j\tau} \quad (80)$$

as shown in Appendix C. At this point, a property of bidimensional FT is recalled. If a function is rotated around the origin of the plane of its variables by a certain angle  $\gamma$ , its FT experiences a rotation by the same angle around the origin of the plane of the conjugate variables. This property can be established in a number of simple ways, one of them being that of resorting to Fourier Bessel expansions. Thus (80) implies:

$$\underline{E}_t(\rho, \phi) = \pm e^{j\pi/N} \underline{H}_t(\rho, \phi - \frac{\pi}{N}) . \quad (81)$$

Consider a self-complementary structure having a center of symmetry (coincident with its feed point). Denote by  $N \geq 2$  the number of "arms" of the antenna (equal to the order of the rotational symmetry). Because of the symmetric structure of the system of Maxwell's equations and of the complementary nature of the boundary conditions, it is known that there exist solutions of the electromagnetic problem of the antenna of the following type: The tangential electric and magnetic field distributions have the same functional form. They are obtained one from the other through a rotation around the center of symmetry of the structure of an angle equal or multiple of one half the angular distance between two adjacent arms, and by a multiplication by a complex constant whose absolute value is equal to  $\eta$ . This means that (81) is a possible solution of the electromagnetic problem of the structure, corresponding therefore to a resistive PWS as defined by (56) or (59). Thus (81) implies that the aperture reactance is necessarily zero.

It is known from experiment that if the self-complementary structure is of a "spiral type", the radiation pattern is almost perfectly polarized no matter what type of spiral is involved (logarithmic, or other). This result is apparently related either to the fast convergence of the series (75) for the radiation pattern or more generally to the predominance in (75) of terms with  $q$  even (or odd) with respect of terms with  $q$  odd (or even). The fact that the structure is a spiral can be expressed in general by saying that in the polar equation of the edge of the arms, the radius vector must be a monotonic function of the angle. How this circumstance affects the convergence property of (75) is an open interesting question.

From a qualitative standpoint, it can be generally asserted that for any rotationally symmetric structure, angular harmonics of the field distribution on the aperture contribute less and less to the PWS in the visible space with the increase of their order. This can be easily established by resorting to Fourier Bessel expansion of both the aperture field and the PWS. On the basis of this reasoning it can be inferred that when  $N$  goes to infinity only one harmonic should predominate in (73) and (75), and the field should be exactly circularly polarized for every direction of propagation. This is in fact the case as shown by (78) and (79).

Finally notice that if (78) holds on a frequency band, then the PWS is resistive on the same frequency band. Since (73) and thus (81) are essentially related to a physical and geometrical property of the aperture, the essential independence of the frequency of the "resistive" character of the PWS is inferred.

## 8. CONCLUDING REMARKS

Two fundamental (strictly interrelated) problem areas concerning the theory of radiating apertures have been investigated in this report by using the Fourier Transform (or, in different terminology) the Plane Wave Spectrum Method.

The first area of study refers to the relationship between the polarization of the radiation pattern and the aperture reactive power. The main results established are summarized by Propositions I and II in Section 4. In Proposition (I), it was asserted that the components of the radiation patterns polarized (for each direction of propagation) in the plane of incidence or orthogonal to it give rise, so to speak, to the capacitive and inductive terms of the reactive power, respectively. In Proposition (II), it was established that an aperture antenna radiating a purely circularly polarized field has reactive power necessarily equal to zero. To obtain these results a somewhat novel formalism was introduced, which allows the determination of the effect on the reactive power of the two "Partial Patterns" in which the radiation pattern of the aperture can be decomposed in an arbitrary way.

The attempt to generalize the result in Proposition (II) spurred an investigation of the second problem area. Determining analytic properties of PWS associated with broadband zero reactance radiating apertures. It was argued that the general (resonance) condition (54) for zero reactive power is, in general strongly dependent upon frequency. This occurs for two basic reasons: the variation of  $E_t(x)$  and therefore  $\underline{E}_t(u)$ , with frequency; and the change of  $k$  (proportional to frequencies), that is, of the radius limit of visible space. Even if  $E_t(x)$  does not vary with frequency (an hypothesis used by Collin, et al., to determine the frequency sensitivity of an aperture admittance<sup>2</sup>), the change of the radius of the visible space will in general upset the balance (51) obtained at resonance. This will not occur, however, if the vector function  $\underline{E}_t$  has certain azimuthal symmetries making the integral for reactive power equal to zero independent of both the radius of the visible space (that is, of the frequency) and of the detailed nature of  $\underline{E}_t$  (provided the symmetry requirements are met). In Proposition III, the analytical conditions were established which the FT of the electric aperture field must satisfy to realize a reactive power density (on the wavenumber plane) with the properties sought. Such condition, as perhaps expected, determines the



azimuthal structure of the PWS only. It does not, however, constrain the functional radial dependence (upon the radius vector  $t$ ) of the coefficients of the angular harmonics of the PWS, which may in fact vary with frequency without affecting the "resistive" character of the PWS. It was recognized that rotationally symmetric self-complementary structure, can support fields whose associated PWS is of the type discussed. For them the condition on the transverse aperture field expressed by (81) is essentially independent of frequency since its realization is related to the self-complementary character of the boundary conditions on the aperture and on the azimuthal number of the excitation, but not on the details of the field distribution.

A final remark is in order. All the discussion has been conducted in terms of admittance properties of the aperture. The antenna input terminal admittance properties depend, however, also upon the nature of the feeding system and may possibly be substantially different. This is true particularly for the broadband resistive properties of self-complementary antennas (always heavily affected by the presence of the cavity, which makes the antenna radiate in a hemisphere only.)

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## Appendix A

### General Expression of Radiation Pattern

The field of an aperture whose transverse dielectric component is  $\underline{E}_t$  is given by the expression:

$$\underline{E}(x, y, z) = -\nabla \times \frac{1}{\pi} \iint_A \frac{\underline{E}_t(\xi, \eta) \exp(-jk \sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2})}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}} d\xi d\eta \quad (A1)$$

as a consequence of one of the equivalence theorems and image principle.<sup>1</sup> By taking the FT of (A1) (a convolution integral) with respect to  $x, y$  and by recalling that the differential operator  $\nabla$  is transformed into the algebraic operator  $-jk\underline{x}$  (and  $\nabla$  into  $\underline{k}$ ), the following expression of the electric field—holding for every point of the hemisphere of radiation—is obtained:

$$\underline{E}(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-j\underline{u} \cdot \underline{x} - jwz} \left( \underline{\epsilon}_t - \frac{\underline{\epsilon}_t \underline{u}}{w} \right) d^2 \underline{u} \quad (A2)$$

where use has been made of a well known integral representation of scalar free space Green's function. After inserting into (A2) the general representation (15)

1. Borgiotti, G. V. (1963) Fourier transforms method in aperture antennas problems, Alta Frequenza 32(No. 1):808-816.

for  $\underline{E}_t$ , the radiation pattern can be obtained by asymptotically evaluating (A2) through a standard application of bidimensional stationary phase method, as discussed in some detail in.<sup>1</sup> The result is Eq. (17) where

$$\vec{F}_{s1}(\theta, \phi) = \left[ \frac{s_1}{2} (\underline{u}_s) \hat{\rho}(\underline{u}_s) \right] \hat{\theta} + \cos \theta \left[ \frac{s_1}{2} (\underline{u}_s) \hat{\Psi}(\underline{u}_s) \right] \hat{\phi} \quad (A3)$$

(the lower or upper subscripts referring to the first and the second partial pattern) where:

$$\underline{u}_s \equiv (k\alpha, k\beta) \quad (A4)$$

is the stationary point.

## Appendix B

### Derivation of the Quadratic Forms for the Complex Power

Because of Maxwell's equation:

$$\underline{H}(x, y, z) = -\frac{1}{jk\eta} \nabla \times \underline{E}(x, y, z) \quad (B1)$$

with  $\underline{E}(x, y, z)$  given by (A1). Taking the FT of (B1) for  $z = 0$ :

$$\underline{H}(\underline{u}) = \underline{H}_t(\underline{u}) + \hat{z} \underline{H}_z(\underline{u}) = \frac{(\underline{u} \cdot \underline{\mathcal{E}}_t \times \hat{z}) \underline{k} - k^2 (\underline{\mathcal{E}}_t \times \hat{z})}{k\eta w} \quad (B2)$$

Thus:

$$\underline{\mathcal{E}}_t = \frac{k^2 \hat{z} \times \underline{\mathcal{E}}_t - (\underline{u} \cdot \hat{z} \times \underline{\mathcal{E}}_t) \underline{u}}{k\eta w} \quad (B3)$$

and:

$$\underline{H}_t^+ \cdot (\hat{z} \times \underline{\mathcal{E}}_t) = \frac{k^2 (\hat{z} \times \underline{\mathcal{E}}_t)^+ \cdot (\hat{z} \times \underline{\mathcal{E}}_t) - (\hat{z} \times \underline{\mathcal{E}}_t \cdot \underline{u})^* (\underline{u} \cdot \hat{z} \times \underline{\mathcal{E}}_t)}{k\eta w} \quad (B4)$$

Since

$$\hat{z}_x \underline{\hat{e}}_t = \hat{\Psi} \underline{\hat{e}}_\rho - \hat{\rho} \underline{\hat{e}}_\Psi \quad (B5)$$

Equation (31) immediately follows.

## Appendix C

### FT Maxwell's Equations in Planar Polar Coordinates

The FT (with respect to  $x, y$ ) of Maxwell's equations are

$$\underline{\mathcal{E}} = \frac{\eta}{k} \underline{\mathcal{H}} \times \underline{k} \quad (C1)$$

$$\underline{\mathcal{H}} = -\frac{1}{\eta k} \underline{\mathcal{E}} \times \underline{k} \quad (C2)$$

with  $\underline{\mathcal{E}} = \underline{\mathcal{E}}_t + \hat{z} \mathcal{E}_z$ . From (C2), taking the radial component on wavenumber plane,

$$\underline{\mathcal{H}}_\rho = -\frac{1}{\eta k} (\underline{\mathcal{E}} \times \underline{k})_\rho = -\frac{w}{k\eta} \mathcal{E}_\psi \quad (C3)$$

From (C2) in a similar way,

$$\mathcal{H}_\psi = \frac{k}{w\eta} \mathcal{E}_\rho \quad (C4)$$

Equations (C3) and (C4) express the relationships between the electric and magnetic vectors for fields TE and TM with respect normal to the aperture, and

show that the two fields are uncoupled.<sup>1</sup> By considering the expression (33) for  $\underline{Y}$ , (80) easily follows from (C3) and C4). (However, Eq. (80) is of course a vector relationship independent of coordinate systems.)

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1. Borgiotti, G. V. (1968) Modal analysis of periodic phased arrays of apertures, Proc. IEEE 56(No. 11):1881-1892 (Appendix I).